1. $\alpha = \dfrac{-1+\sqrt{3}i}{2}$ 일 때, $\alpha + \alpha^2 + \cdots + \alpha^{14}$ 의 값은?

① -1 ② $-\dfrac{1}{2}$ ③ 0 ④ $\dfrac{1}{2}$ ⑤ 1

해설

$\alpha = \dfrac{-1+\sqrt{3}i}{2}$ 에서 $2\alpha + 1 = \sqrt{3}i$

양변을 제곱해서 정리하면 $\alpha^2 + \alpha + 1 = 0$

$(\alpha - 1)(\alpha^2 + \alpha + 1) = 0, \alpha^3 = 1$

$\therefore\ \alpha^{3k+1} = \alpha, \alpha^{3k+2} = \alpha^2, \alpha^{3k} = 1$

(준식) $= (\alpha + \alpha^2 + 1) + (\alpha + \alpha^2 + 1) +$

$\cdots + (\alpha + \alpha^2 + 1) + \alpha + \alpha^2$

$= \alpha + \alpha^2$

$= -1$

($\because\ \alpha^2 + \alpha + 1 = 0$)

2. $\alpha = \dfrac{-1+\sqrt{3}i}{2}$ 일 때, 다음 보기 중 옳은 것을 모두 고른 것은? (단, $\bar{z}$ 는 z 의 켤레복소수)

㉠ $\alpha^2+\alpha+1=0$
㉡ $1+\alpha+\alpha^2+\cdots+\alpha^{15}=1$
㉢ $z=\dfrac{\alpha+3}{2\alpha+1}$ 일 때, $z\bar{z}=\dfrac{7}{3}$

① ㉠ ② ㉠ , ㉡ ③ ㉠ , ㉢
④ ㉡ , ㉢ ⑤ ㉠ , ㉡ , ㉢

해설

㉠ : $\alpha = \dfrac{-1+\sqrt{3}i}{2}$, $2\alpha+1=\sqrt{3}i$

양변을 제곱해서 정리하면 $\alpha^2+\alpha+1=0$

㉡ : $(\alpha-1)(\alpha^2+\alpha+1)=0$, $\alpha^3=1$

$1+\alpha+\alpha^2+\cdots+\alpha^{15}$

$=1+\alpha+\alpha^2+\alpha^3(1+\alpha+\alpha^2)+\cdots+\alpha^{15}=\alpha^{15}$

$=(\alpha^3)^5=1$ ($\because \alpha^2+\alpha+1=0$)

㉢ : $\bar{\alpha} = \dfrac{-1-\sqrt{3}i}{2}$, $\alpha+\bar{\alpha}=-1, \alpha\bar{\alpha}=1$

$z=\dfrac{\alpha+3}{2\alpha+1}$, $\bar{z}=\dfrac{\bar{\alpha}+3}{2\bar{\alpha}+1}$

$z\bar{z}=\dfrac{\alpha\bar{\alpha}+3(\alpha+\bar{\alpha})+9}{4\alpha\bar{\alpha}+2(\alpha+\bar{\alpha})+1}=\dfrac{1-3+9}{4-2+1}=\dfrac{7}{3}$

해설

㉢ 이 성립함을 다음과 같이 직접 계산할 수 있다.

$\alpha = \dfrac{-1+\sqrt{3}i}{2}$

$\Rightarrow 2\alpha+1=\sqrt{3}i, \alpha+3=\dfrac{5+\sqrt{3}i}{2}$

$\therefore \dfrac{\alpha+3}{2\alpha+1}=\dfrac{5+\sqrt{3}i}{2\sqrt{3}i}$

$=-\dfrac{5i-\sqrt{3}}{2\sqrt{3}}$

$z\cdot\bar{z}=\dfrac{\sqrt{3-5i}}{2\sqrt{3}}\times\dfrac{\sqrt{3+5i}}{2\sqrt{3}}=\dfrac{7}{3}$

3. $\alpha = \dfrac{-1+\sqrt{3}i}{2}$ 일 때, $\alpha^3+2\alpha^2+2\alpha+5$ 의 값을 구하면?

① 3　② 4　③ 5　④ 6　⑤ 7

해설

$$\alpha = \frac{-1+\sqrt{3}i}{2}$$
$$2\alpha = -1+\sqrt{3}i$$
$$2\alpha+1 = \sqrt{3}i$$
양변을 제곱하여 정리하면
$$\alpha^2+\alpha+1=0$$
$$\alpha^3+2\alpha^2+2\alpha+5$$
$$= \alpha(\alpha^2+\alpha+1)+(\alpha^2+\alpha+1)+4$$
$$= 4$$

해설

$\alpha^2+\alpha+1=0$ 을 얻은 후 $\alpha^3+2\alpha^2+2\alpha+5$ 를 $\alpha^2+\alpha+1$ 로 나누면
$$\alpha^3+2\alpha^2+2\alpha+5$$
$$= (\alpha^2+\alpha+1)(\alpha+1)+4$$
$$= 4\ (\because\ \alpha^2+\alpha+1=0)$$

4. $a = \dfrac{-1-\sqrt{3}i}{2}$ 일 때, $a^5 + a^3 - 1$ 의 값을 구하면? (단, $i = \sqrt{-1}$)

① $\dfrac{1-\sqrt{3}i}{2}$　② 0　③ 1

④ $\dfrac{-1+\sqrt{3}i}{2}$　⑤ $-1+\sqrt{3}i$

해설

$a = \dfrac{-1-\sqrt{3}i}{2}$

$2a+1 = -\sqrt{3}i$ 의 양변을 제곱하면,

$4a^2 + 4a + 1 = -3 \Rightarrow a^2 + a + 1 = 0$

양변에 $a-1$ 를 곱하면

$(a-1)(a^2+a+1) = 0 \Leftrightarrow a^3 - 1 = 0$

$\therefore\ a^3 = 1$

$(\text{준식}) = a^3a^2 + a^3 - 1$

$= a^2$

$= -a - 1 (\because a^2 + a + 1 = 0)$

$= \dfrac{-1+\sqrt{3}i}{2}$