1. 수열 $\frac{1}{1+\sqrt{2}}, \frac{1}{\sqrt{2}+\sqrt{3}}, \frac{1}{\sqrt{3}+\sqrt{4}}, \cdots$ 의 제 15항까지의 합은?

① $\sqrt{14}-1$　② $\sqrt{15}-1$　[✓ ③ 3]

④ $\sqrt{15}+1$　⑤ 5

해설

$\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{15}+\sqrt{16}}$

$=\sum_{k=1}^{15}\frac{1}{\sqrt{k}+\sqrt{k+1}}$

$=\sum_{k=1}\frac{\sqrt{k}-\sqrt{k+1}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k}-\sqrt{k+1})}$

$=-\sum_{k=1}^{15}(\sqrt{k}-\sqrt{k+1})$

$=-\left\{(1-\sqrt{2})+(\sqrt{2}-\sqrt{3})+\cdots\right\}$

$-\left\{(\sqrt{15}-\sqrt{16})\right\}$

$=-(1-\sqrt{16})=\sqrt{16}-1=4-1=3$

2. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$ 의 값은?

① $\frac{1}{n+1}$　② $\frac{2n}{n+1}$　[✓ ③ $\frac{n}{2n+1}$]

④ $\frac{n}{n+2}$　⑤ $\frac{2n}{2n+1}$

해설

준 식) $= \frac{1}{2}\sum_{k=1}^{n}\left\{\frac{1}{2k-1} - \frac{1}{2k+1}\right\}$

$= \frac{1}{2} \cdot \left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\right\} + \cdots +$

$\frac{1}{2}\left\{\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$

$= \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$

$= \frac{n}{2n+1}$

3. $\sum_{k=1}^{n} \frac{1}{4k^2-1}$ 의 값은?

① $\frac{1}{n+1}$ ② $\frac{n}{n+1}$ ③ $\frac{2n}{n+1}$

[✓ ④ $\frac{n}{2n+1}$] ⑤ $\frac{2n}{2n+3}$

해설

(주어진 식) $= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$

$\frac{1}{2} \left\{ \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) \right\}$

$+ \cdots + \frac{1}{2} \left\{ \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right\}$

$= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) = \frac{n}{2n+1}$

4. 수열의 합 $S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$ 을 간단히 하면? (단, $x \neq 1$)

① $S = \dfrac{n(1-x^n)}{2}$　　② $S = \dfrac{1-x^n}{2}$

③ $S = \dfrac{1-x^n}{2} - \dfrac{2x^n}{x}$　　④ $S = \dfrac{1-x^n}{1+x} - \dfrac{1-x^n}{(1-x)^2}$

[✓ ⑤] $S = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{1-x}$

해설

등차수열과 등비수열의 곱으로 이루어진 멱급수의 형태이므로 양변에 x를 곱하여 변끼리 빼면

$$\begin{array}{rl} S = & 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} \\ -)\ xS = & \quad\ x + 2x^2 + 3x^3 + \cdots + (n-1)x^n + nx^n \\ \hline (1-x)S = & 1 + x + x^2 + x^3 + \cdots + x^n - n \cdot x \end{array}$$

$= \dfrac{1(1-x^n)}{1-x} - n \cdot x^n$

$\therefore S = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{1-x}$

5. 다음 수열의 합을 구하여라.

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 9 \cdot 2^9$$

▶ **답 :**

▷ **정답 :** 8194

해설

$S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 9 \cdot 2^9 \cdots$ ㉠

$2S = 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + 8 \cdot 2^9 + 9 \cdot 2^{10} \cdots$ ㉡

이므로 ㉠-㉡을 하면

$$\begin{aligned} -S &= \frac{2(2^9 - 1)}{2 - 1} - 9 \cdot 2^{10} \\ &= 2 \cdot 2^9 - 2 - 9 \cdot 2^{10} \\ &= 2 \cdot 2^9 - 18 \cdot 2^9 - 2 \\ &= -16 \cdot 2^9 - 2 \end{aligned}$$

$\therefore S = 2^{13} + 2 = 1024 \times 8 + 2 = 8194$

6. 자연수 n 이하의 모든 수의 곱을 $n!$로 나타낸다. 예를 들어 $5! = 5 \times 4 \times 3 \times 2 \times 1$이다. 이때, $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{10}{11!}$의 값은?

① $\frac{9}{10!}$ ② $\frac{10}{11!}$ ③ $1 - \frac{1}{10!}$

[✓ ④ $1 - \frac{1}{11!}$] ⑤ $1 - \frac{1}{12!}$

해설

일반항을 a_n이라 하면 $a_n = \frac{n}{(n+1)!}$이다.

여기서 분자를 변형하면 부분분수의 꼴로 바꿀 수 있다.

즉, $\frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$

$= \frac{1}{n!} - \frac{1}{(n+1)!}$

$\therefore \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2}{11!}$

$= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \cdots + \left(\frac{1}{10!} - \frac{1}{11!}\right)$

$= 1 - \frac{1}{11!}$